module-05

# Numerical Solution of Second order Ordinary differential equations

## Introduction:

The given Second order ODE with two initial Conditions will reduce to two first order simultaneous ODEs which can be solved.

Let $y'' = g(x, y, y')$ with the initial Conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$ be the second order DE.

Now, Let $y' = \frac{dy}{dx} = z$.

This gives $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx}$

The given Second order DE assumes the form: $\frac{dz}{dx} = g(x, y, z)$ with the conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ where $y_0'$ is denoted by $z_0$.

Hence, we now have two first order simultaneous ODEs.

(1) $\frac{dy}{dx} = z$ and (2) $\frac{dz}{dx} = g(x, y, z)$ with $y(x_0) = y_0$ and $z(x_0) = z_0$

Taking $f(x, y, z) = z$, we now have the following system of equations for solving

$\frac{dy}{dx} = f(x, y, z)$, $\frac{dz}{dx} = g(x, y, z)$;

$y(x_0) = y_0$ and $z(x_0) = z_0$

Runge - kutta Method

we have to compute $y(x_0+h)$ and if required $y'(x_0+h) = z(x_0+h)$.

we need to first compute the following

$k_1 = hf(x_0, y_0, z_0)$ ; $l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$;

$l_2 = hg(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$;

$l_3 = hg(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$;

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

The required

$y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and $y'(x_0+h) = z(x_0+h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

Problems

P.T.O

① Given $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, $y(0) = 1$, $y'(0) = 0$. Evaluate $y(0.1)$ using Runge kutta method of order 4.

Solⁿ: By data

$$\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1, \quad y(0) = 1 \text{ and } y'(0) = 0$$

$$y(0.1) = ?$$

$y = 1$, $y' = 0$ at $x = 0$

$y' = \frac{dy}{dx} = z$

$y'' = \frac{d^2y}{dx^2} = z'$

putting, $\frac{dy}{dx} = z$

and Differentiating, w.r. to $x$ we obtain

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

So that given equation becomes

$$\frac{dz}{dx} - x^2 z - 2xy = 1$$

hence, we have a system of equations

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = 1 + 2xy + x^2 z \quad \text{where}$$

$y(0) = 1$, $y'(0) = 0$, $y'_0 = 0$

$y_0 = 1$, $z_0 = 0$, $x_0 = 0$

Let, $f(x, y, z) = z$, $g(x, y, z) = 1 + 2xy + x^2 z$

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$ ($y'_0 = 0$) and let us take

$h = 0.1$

$x = x_0 + h$

$0.1 = 0 + h$

$h = 0.1$

we shall first find the following

$K_1 = hf(x_0, y_0, z_0)$

$K_1 = (0.1) f(0, 1, 0)$

$K_1 = (0.1)(0)$

$\underline{\underline{K_1 = 0}}$

$l_1 = hg(x_0, y_0, z_0)$

$l_1 = (0.1) g(0, 1, 0)$

$l_1 = (0.1)[1 + 2(0)(1) + (0)(0)]$

$l_1 = (0.1)(1)$

$l_1 = 0.1$ //

$K_2 = hf(x_0 + h/2,\ y_0 + K_1/2,\ z_0 + l_1/2)$

$= (0.1) f\left(0 + \frac{0.1}{2},\ 1 + 0/2,\ 0 + \frac{0.1}{2}\right)$

$= (0.1) f(0.05,\ 1,\ 0.05)$

$= (0.1)(0.05)$

$\underline{\underline{K_2 = 0.005}}$

$l_2 = hg(x_0 + h/2,\ y_0 + K_1/2,\ z_0 + l_1/2)$

$= (0.1) g(0 + 0.1/2,\ 1 + 0/2,\ 0 + 0.1/2)$

$= (0.1) g(0.05,\ 1,\ 0.05)$

$= (0.1)[1 + 2(0.05)(1) + (0.05)^2(0.05)]$

$\underline{\underline{l_2 = 0.110012}}$

$K_3 = hf(x_0 + h/2,\ y_0 + K_2/2,\ z_0 + l_2/2)$

$= (0.1) f\left(0 + 0.1/2,\ 1 + \frac{0.005}{2},\ 0 + \frac{0.110012}{2}\right)$

$= (0.1) f(0.05,\ 1.0025,\ 0.055)$

$= (0.1)(0.055)$

$= 0.0055$ //

$l_3 = hg(x_0 + h/2,\ y_0 + K_2/2,\ z_0 + l_2/2)$

$= (0.1) g\left(0 + 0.1/2,\ 1 + \frac{0.005}{2},\ 0 + \frac{0.110012}{2}\right)$

$= (0.1) g(0.05,\ 1.0025,\ 0.055)$

$= (0.1)[1 + 2(0.05)(1.0025) + (0.05)^2(0.055)]$

$= 0.110038$ //

$K_4 = hf(x_0+h,\ y_0+K_3,\ z_0+l_3)$

$= (0.1)f(0+0.1,\ 1+0.0055,\ 0+0.110038)$

$= (0.1)f(0.1,\ 1.0055,\ 0.110038)$

$= (0.1)(0.110038)$

$K_4 = 0.011$

$l_4 = hg(x_0+h,\ y_0+K_3,\ z_0+l_3)$

$= (0.1)g(0+0.1,\ 1+0.0055,\ 0+0.110038)$

$= (0.1)g(0.1,\ 1.0055,\ 0.110038)$

$= (0.1)[1+2(0.1)(1.0055)+(0.1)^2(0.110038)]$

$l_4 = 0.12022$

we have to find $y(0.1)$

$y(x_0+h) = y_0 + 1/6\,(K_1+2K_2+2K_3+K_4)$

$y(0.1) = 1 + 1/6\,(0+2(0.005)+2(0.0055)+0.011)$

$y(0.1) = 1.0053$

NOTE: $Z(x_0+h) = Z_0 + 1/6\,(l_1+2l_2+2l_3+l_4)$

$Z(0+0.1) = 0 + 1/6\,(0.1+2(0.11012)+2(0.110038)+0.12022)$

See In this problem they are not asked to find $Z(0.1)$

$Z(0.1) = 0.110089$

② By Runge-Kutta method, solve $\dfrac{d^2y}{dx^2} = x\left(\dfrac{dy}{dx}\right)^2 - y^2$ for $x = 0.2$ correct to four decimal places, using the initial conditions $y=1$ and $y'=0$ when $x=0$

June & Dec 2017

Sol: By data $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$, $y_0 = 1$, $z_0 = 0$, $x_0 = 0$

$y(0.2) = ?$

put $\frac{dy}{dx} = z$

D. w. r. to $x$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Now given equation becomes

$y' = 0$

$$\frac{dz}{dx} = xz^2 - y^2 \quad \text{with } y_0 = 1, \ z_0 = 0 \text{ at } x_0 = 0$$

hence we have

$\frac{dy}{dx} = z$ and $\frac{dz}{dx} = xz^2 - y^2$

Let $f(x, y, z) = z$, $g(x, y, z) = xz^2 - y^2$,

$x_0 = 0$, $y_0 = 1$, $z_0 = 0$

$x_0 + h = 0.2$

$0 + h = 0.2$

$h = 0.2$

we shall find the following

$K_1 = hf(x_0, y_0, z_0)$

$K_1 = (0.2) f(0, 1, 0)$

$= (0.2)(0)$

$K_1 = 0$

$l_1 = hg(x_0, y_0, z_0)$

$= (0.2) g(0, 1, 0)$

$= (0.2)[(0)(0)^2 - 1^2]$

$= (0.2)[-1]$

$l_1 = -0.2$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + 0/2, 0 + \frac{(-0.2)}{2}\right)$

$= (0.2) f(0.1, 1, -0.1)$

$= (0.2)(-0.1)$

$K_2 = -0.02$

$$l_2 = hg(x_0+h/2,\ y_0+k_1/2,\ z_0+l_1/2)$$

$$=(0.2)g\left(0+\frac{0.2}{2},\ 1+0/2,\ 0+\frac{(-0.2)}{2}\right)$$

$$=(0.2)g(0.1,\ 1,\ -0.1)$$

$$=(0.2)[(0.1)(-0.1)^2-(1)^2]$$

$$=(0.2)[-0.999]$$

$$l_2 = -0.1998$$

$$k_3 = hf(x_0+h/2,\ y_0+k_2/2,\ z_0+l_2/2)$$

$$=(0.2)f\left(0+0.2/2,\ 1+\frac{(-0.02)}{2},\ 0+\frac{(-0.1998)}{2}\right)$$

$$=(0.2)f(0.1,\ 0.99,\ -0.0999)$$

$$=(0.2)(-0.0999)$$

$$k_3 = -0.01998$$

$$l_3 = hg(x_0+h/2,\ y_0+k_2/2,\ z_0+l_2/2)$$

$$=(0.2)g\left(0+0.2/2,\ 1+\frac{(-0.02)}{2},\ 0+\frac{(-0.1998)}{2}\right)$$

$$=(0.2)g(0.1,\ 0.99,\ 0.0999)$$

$$=(0.2)[(0.1)(0.0999)^2-(0.99)^2]$$

$$=(0.2)[-0.979101999]$$

$$l_3 = -0.1958$$

$$k_4 = hf(x_0+h,\ y_0+k_3,\ z_0+l_3)$$

$$=(0.2)f(0+0.2,\ 1+(-0.01998),\ 0+(-0.1958))$$

$$=(0.2)f(0.2,\ 0.98002,\ -0.1958)$$

$$=(0.2)(-0.1958)$$

$$k_4 = -0.03916$$

$l_4 = hg(x_0+h,\ y_0+k_3,\ z_0+l_3)$

$= (0.2)\, g(0+0.2,\ 1-0.01998,\ 0-0.1958)$

$= (0.2)\, g(0.2,\ 0.98002,\ -0.1958)$

$= (0.2)\left((0.2)(-0.1958)^2 - (0.98002)^2\right)$

$l_4 = -0.19055$

we need to find $y(0.2)$

$\therefore y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y(0.2) = 1 + \frac{1}{6}(0 + 2(-0.02) + 2(-0.01998) - 0.03916)$

$\boxed{y(0.2) = 0.9801}$

NOTE: $Z(x_0+h) = Z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$Z(0.2) = 0 + \frac{1}{6}\left(-0.2 + 2(-0.1998) + 2(-0.1958) + (-0.19055)\right)$

$\boxed{Z(0.2) = -0.1969}$

③ Compute $y(0.1)$ given $\frac{d^2y}{dx^2} = y^3$ and $y=10$, $\frac{dy}{dx} = 5$ at $x=0$ by Runge kutta method of fourth order.

Sol$^n$: By data $\frac{d^2y}{dx^2} = y^3$, $y_0 = 10$, $\frac{dy}{dx} = y' = 5$ at $x_0 = 0$

$y(0.1) = ?$

put $\frac{dy}{dx} = Z$

D. w. r. to $x$

$\frac{d^2y}{dx^2} = \frac{dZ}{dx}$

given equation becomes

$\frac{dZ}{dx} = y^3$

hence we have system of equations

$\frac{dy}{dx} = Z$ and $\frac{dZ}{dx} = y^3$

$x_0 + h = 0.1$

$h = 0.1 - x_0$

$h = 0.1 - 0$

$h = 0.1$

Let $f(x,y,z) = z$ and $g(x,y,z) = z^3$

with $x_0 = 0$, $y_0 = 10$, $z_0 = 5$

we shall first find the following

$k_1 = hf(x_0, y_0, z_0)$

$k_1 = (0.1) f(0, 10, 5)$

$= (0.1)(5)$

$k_1 = (0.5)$

$l_1 = hg(x_0, y_0, z_0)$

$= (0.1) g(0, 10, 5)$

$= (0.1)(10)^3$

$= 100$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$= (0.1) f\left(0 + \frac{0.1}{2}, 10 + \frac{0.5}{2}, 5 + \frac{100}{2}\right)$

$= (0.1) f(0.05, 10.25, 55)$

$= (0.1)(55)$

$= 5.5$

$l_2 = hg(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$= (0.1) g(0.05, 10.25, 55)$

$= (0.1)(10.25)^3$

$= 107.68$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= (0.1) f\left(0 + 0.1/2, 10 + 5.5/2, 5 + \frac{107.68}{2}\right)$

$= (0.1) f(0.05, 12.75, 58.84)$

$= (0.1)(58.84)$

$= 5.884$

$l_3 = hg(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= (0.1) g(0.05, 12.75, 58.84)$

$= (0.1)(12.75)^3$

$= 207.267$

$k_4 = hf(x_0+h, y_0+k_3, z_0+l_3)$

$= (0.1)f(0+0.1, 10+5.884, 5+207.267)$

$= (0.1)f(0.1, 15.884, 212.267)$

$= (0.1)(212.267)$

$= 21.2267$

$l_4 = hg(x_0+h, y_0+k_3, z_0+l_3)$

$= (0.1)g(0.1, 15.884, 212.267)$

$= (0.1)(15.884)^3$

$= 400.75$

we have to find $y(0.1)$

$y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 10 + \frac{1}{6}(0.5 + 2(5.5) + 2(5.884) + 21.2267)$

$y(0.1) = \underline{17.4157}$

(4) Given $y'' - xy' - y = 0$ with the initial conditions $y(0) = 1$, $y'(0) = 0$, compute $y(0.2)$ and $y'(0.2)$ using fourth order Runge Kutta method.

June 2018

Soln: By data $y'' - xy' - y = 0$

$y(0) = 1, \; y'(0) = 0$

$y(0.2) = ?$ and $y'(0.2) = ?$

put $\frac{dy}{dx} = z$

D. w. r. to $x$

$\frac{d^2y}{dx^2} = \frac{dz}{dx}$

Given equation becomes

$y'' - xy' - y = 0$

$\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$

$\frac{dz}{dx} - xz - y = 0$

$\frac{dz}{dx} = y + xz$

Now, we have system of equations

$\frac{dy}{dx} = z$ and $\frac{dz}{dx} = y + xz$

with $x_0 = 0, y_0 = 1, z_0 = 0$ $y'_0 = 0$

Let $f(x, y, z) = z$ and $g(x, y, z) = y + xz$

with $x_0 = 0, y_0 = 1, z_0 = 0$

we shall find the following

$x_0 + h = 0.2$

$h = 0.2 - x_0$

$h = 0.2 - 0$

$h = 0.2$

$K_1 = hf(x_0, y_0, z_0)$

$K_1 = 0.2 f(0, 1, 0)$

$= 0.2(0)$

$K_1 = 0$

$l_1 = hg(x_0, y_0, z_0)$

$l_1 = (0.2) g(0, 1, 0)$

$= (0.2)[1 + (0)(0)]$

$= (0.2)(1)$

$= 0.2$

$K_2 = hf(x_0 + h/2, y_0 + K_1/2, z_0 + l_1/2)$

$= (0.2) f\left(0 + \frac{0.2}{2}, 1 + 0/2, 0 + \frac{0.2}{2}\right)$

$= (0.2) f(0.1, 1, 0.1)$

$= (0.2)(0.1)$

$= 0.02$

$l_2 = hg(x_0 + h/2, y_0 + k_1/2, z_0 + l_1/2)$

$= (0.2)g(0.1, 1, 0.1)$

$= (0.2)[1 + (0.1)(0.1)]$

$= 0.202$ //

$k_3 = hf(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= (0.2)f\left(0 + \frac{0.2}{2}, 1 + \frac{0.02}{2}, 0 + \frac{0.202}{2}\right)$

$= (0.2)f(0.1, 1.01, 0.101)$

$= (0.2)(0.101)$

$k_3 = 0.0202$

$l_3 = hg(x_0 + h/2, y_0 + k_2/2, z_0 + l_2/2)$

$= (0.2)g(0.1, 1.01, 0.101)$

$= (0.2)[1.01 + (0.1)(0.101)]$

$l_3 = 0.20402$ //

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.2)f(0 + 0.2, 1 + 0.0202, 0 + 0.20402)$

$= (0.2)f(0.2, 1.0202, 0.20402)$

$= (0.2)(0.20402)$

$= 0.0408$

$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.2)g(0.2, 1.0202, 0.20402)$

$= (0.2)[1.0202 + (0.2)(0.20402)]$

$l_4 = 0.2122$

Now we have to find $y(0.2)$ and $y'(0.2)$ or $z(0.2)$

we have

$y(x_0+h) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$y(0.2) = 1 + \frac{1}{6}(0 + 2(0.02) + 2(0.0202) + 0.0408)$

$y(0.2) = 1.0202$ //

Then we have to find $y'(0.2)$ or $z(0.2)$

$z(x_0+h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= 0 + \frac{1}{6}(0.2 + 2(0.202) + 2(0.20402) + 0.21$

$z(0.2) = 0.20404$ or $y'(0.2) = 0.20404$

Thus $y(0.2) = 1.0202$ and $y'(0.2) = 0.20404$

(5) Obtain the value of $x$ and $\frac{dx}{dt}$ when $t = 0.1$ given that $x$ satisfies the equation $\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$ and $x = 3$, $\frac{dx}{dt} = 0$ when $t = 0$ initially. Use 4th order Runge Kutta method.

No Need

Soln: By data

$\frac{d^2x}{dt^2} = t\frac{dx}{dt} - 4x$, $x = 3, \frac{dx}{dt} = 0, t = 0$

put $\frac{dx}{dt} = y$

D.w.r. to $t$

$\frac{d^2x}{dt^2} = \frac{dy}{dt}$

given equation becomes

$\frac{dy}{dt} = ty - 4x$

hence we have system of equations

$\frac{dx}{dt} = y$ and $\frac{dy}{dt} = ty - 4x$

with $t_0 = 0, x_0 = 3, \frac{dx}{dt} = y_0 = 0$

$t_0 + h = 0.1$

$h = 0.1 - t_0$

$h = 0.1 - 0$

$h = 0.1$

let $f(t, x, y) = y$ and $g(t, x, y) = ty - 4x$

with $t_0 = 0$, $x_0 = 3$, $y_0 = 0$

we shall find the following

$k_1 = hf(t_0, x_0, y_0)$

$= (0.1) f(0, 3, 0)$

$= (0.1)(0)$

$= 0$

$l_1 = hg(t_0, x_0, y_0)$

$= (0.1) g(0, 3, 0)$

$= (0.1)[(0)(0) - 4(3)]$

$l_1 = -1.2$

$k_2 = hf(t_0 + h/2, x_0 + k_1/2, y_0 + l_1/2)$

$= (0.1) f(0.05, 3, -0.6)$

$= (0.1)(-0.6)$

$= -0.06$

$l_2 = hg(t_0 + h/2, x_0 + k_1/2, y_0 + l_1/2)$

$= (0.1) g(0.05, 3, -0.6)$

$= (0.1)[(0.05)(-0.6) - 12]$

$= -1.203$

$k_3 = hf(t_0 + h/2, x_0 + k_2/2, y_0 + l_2/2)$

$= (0.1) f(0.05, 2.97, -0.6015)$

$k_3 = (0.1)(-0.6015)$

$\underline{\underline{k_3}} = -0.06015$

$l_3 = (0.1)[(0.05)(-0.6015) - 4 \times 2.97]$

$l_3 = \underline{\underline{-1.191}}$

$k_4 = hf(t_0+h, x_0+k_3, y_0+l_3)$

$= (0.1) f(0.1, 2.93985, -1.191)$

$= (0.1)(-1.191)$

$k_4 = \underline{\underline{-0.1191}}$

$l_4 = hg(t_0+h, x_0+k_3, y_0+l_3)$

$= (0.1) g(0.1, 2.93985, -1.191)$

$= (0.1)[(0.1)(-1.191) - 4 \times 2.93985]$

$l_4 = \underline{\underline{-1.18785}}$

$x(t_0+h) = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$\underline{\underline{x(0.1)}} = 2.9401$

$y(t_0+h) = y_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$y(0.1) = \underline{\underline{-1.196}}$

# Milne's Method

Method to solve the ODE $y'' = g(x, y, y')$ given a set of four initial values for $y$ and $y'$.

① Consider $y'' = g(x, y, y')$ with initial condition $y(x_0) = y_0$ and $y'(x_0) = y_0'$

② put $y' = \frac{dy}{dx} = z$

D. w. r. to $x$

$$y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$$

$$\therefore y'' = z'$$

the given differential equation becomes

$z' = g(x, y, z)$ with initial condition

$y(x_0) = y_0$ and $z(x_0) = z_0$

③ The given data's are set of $x$ values i.e $x_0, x_1, x_2, x_3, x_4$

set of $y$ values $y_0, y_1, y_2, y_3,$

set of $y'$ or $z$ values $z_0, z_1, z_2, z_3$

④ First apply predictor formula to find $y_4^{(P)}$ and $z_4^{(P)}$

where $$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

⑤ we compute $z_4' = g(x_4, y_4, z_4)$ and then apply corrector formula

where $$y_4^{(C)} = y_2 + h/3\,(z_2 + 4z_3 + z_4)$$

$$z_4^{(C)} = z_2 + h/3\,(z_2' + 4z_3' + z_4')$$

⑥ For better accuracy apply corrector formula repeatedly

Problems

① Apply milne's method to solve $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ given the following table of initial values compute $y(0.4)$

June 2018

| $x$ | 0 | 0.1 | 0.2 | 0.3 |
|---|---|---|---|---|
| $y$ | 1 | 1.1103 | 1.2427 | 1.399 |
| $y'$ | 1 | 1.2103 | 1.4427 | 1.699 |

Sol$^n$: By data $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$

put $y' = \frac{dy}{dx} = Z$

we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$ $\therefore$ $y'' = z'$

Given equation becomes

$$\frac{dz}{dx} = 1 + Z$$

or $Z' = 1 + Z$

| $x$ | $x_0 = 0$ | $x_1 = 0.1$ | $x_2 = 0.2$ | $x_3 = 0.3$ |
|---|---|---|---|---|
| $y$ | $y_0 = 1$ | $y_1 = 1.1103$ | $y_2 = 1.2427$ | $y_3 = 1.399$ |
| $y' = Z$ | $Z_0 = 1$ | $Z_1 = 1.2103$ | $Z_2 = 1.4427$ | $Z_3 = 1.699$ |
| $y'' = Z' = 1 + Z$ | $Z_0' = 1 + Z_0$ <br> $Z_0' = 1 + 1 = 2$ | $Z_1' = 1 + Z_1 = 1 + 1.2103 = 2.2103$ | $Z_2' = 1 + Z_2 = 1 + 1.2427 = 2.4427$ | $Z_3' = 1 + Z_3 = 1 + 1.399 = 2.699$ |

we first consider milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2Z_1 - Z_2 + 2Z_3)$$

$$y_4^{(P)} = 1 + \frac{4(0.1)}{3}\left[2(1.2103) - 1.4427 + 2(1.699)\right]$$



$$y_4^{(P)} = 1.58345$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

$$z_4^{(P)} = 1 + \frac{4(0.1)}{3}\left(2(2.2103) - (2.4427) + 2(2.699)\right)$$

$$z_4^{(P)} = 1.98345$$

$$z_4' = 1 + z_4$$

$$z_4' = 1 + 1.98345 \Rightarrow z_4' = 2.98345$$

Now, we consider milne's corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$= 1.2427 + \frac{0.1}{3}\left(1.4427 + 4(1.699) + 1.98345\right)$$

$$= 1.58344$$

$$z_4^{(c)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

$$= 1.4427 + \frac{0.1}{3}\left(2.4427 + 4(2.699) + 2.98345\right)$$

$$= 1.98344$$

Substituting again in corrector formula

$$y_4^{(c)} = 1.2427 + \frac{0.1}{3}\left[1.4427 + 4(1.699) + 1.98344\right]$$

$$y_4^{(c)} = 1.58344$$

Thus $y(0.4) = 1.58344$

$= \cancel{1.4427 + 0.1/3\,(2.44\ldots}$

$z_4^{(c)} \cancel{= 1.98344}$

$\therefore y(0.4) = 1.58344$ and $z(0.4) = 1.98344$

② Apply milne's method to compute $y(0.8)$ given that $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$ and the following table of initial values.

| | | | | |
|---|---|---|---|---|
| $x$ | 0 | 0.2 | 0.4 | 0.6 |
| $y$ | 0 | 0.02 | 0.0795 | 0.1762 |
| $y'$ | 0 | 0.1996 | 0.3937 | 0.5689 |

Apply corrector formula twice in preventing the value of $y$ at $x = 0.8$

Sol^n: Given $\frac{d^2y}{dx^2} = 1 - 2y\frac{dy}{dx}$

put $y' = \frac{dy}{dx} = z$

D. w. r. to $x$

we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$ $\therefore y'' = z'$

given equation becomes

$$z' = 1 - 2yz$$

| $x$ | $x_0 = 0$ | $x_1 = 0.2$ | $x_2 = 0.4$ | $x_3 = 0.6$ |
|---|---|---|---|---|
| $y$ | $y_0 = 0$ | $y_1 = 0.02$ | $y_2 = 0.0795$ | $y_3 = 0.1762$ |
| $y' = z$ | $z_0 = 0$ | $z_1 = 0.1996$ | $z_2 = 0.3937$ | $z_3 = 0.5689$ |
| $y'' = z'$ $= 1 - 2yz$ | $z_0' = 1 - 2y_0z_0$ $= 1 - 2(0)(0)$ $z_0' = 1$ | $z_1' = 1 - 2y_1z_1$ $z_1' = 0.992$ | $z_2' = 1 - 2y_2z_2$ $z_2' = 0.9374$ | $z_3' = 1 - 2y_3z_3$ $z_3' = 0.7995$ |

we first consider milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$= 0 + \frac{4(0.2)}{3}\left(2(0.1996) - (0.3937) + 2(0.5689)\right)$$

$$y_4^{(P)} = 0.30488$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left(2z_1' - z_2' + 2z_3'\right)$$

$$= 0 + \frac{4(0.2)}{3}\left(2(0.992) - (0.9374) + 2(0.7995)\right)$$

$$z_4^{(P)} = \underline{\underline{0.70549}}$$



$$z_4' = 1 - 2y_4 z_4$$

$$= 1 - 2(0.30488)(0.70549)$$

$$= \underline{\underline{0.56982}}$$

Now we consider milne's corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}\left(z_2 + 4z_3 + z_4\right)$$

$$= 0.0795 + \frac{0.2}{3}\left(0.3937 + 4(0.5689) + 0.70549\right)$$

$$= \underline{\underline{0.30448}}$$

$$z_4^{(c)} = z_2 + \frac{h}{3}\left(z_2' + 4z_3' + z_4'\right)$$

$$= 0.3937 + \frac{0.2}{3}\left(0.9374 + 4(0.7995) + 0.56982\right)$$

$$= \underline{\underline{0.70738}}$$

Substituting the appropriate value in corrector formula

$$y_4^{(c)} = 0.0795 + \frac{0.2}{3}\left(0.3937 + 4(0.5689) + 0.70738\right)$$

$$= \underline{\underline{0.3046}}$$

Thus $\underline{\underline{y(0.8) = 0.3046}}$

③ Obtain the solution of the equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$ by computing the value of the dependent variable corresponding to the value 1·4 of the independent variable by applying milne's method using the following data.

| | | | | |
|---|---|---|---|---|
| $x$ | 1 | 1·1 | 1·2 | 1·3 |
| $y$ | 2 | 2·2156 | 2·4649 | 2·7514 |
| $y'$ | 2 | 2·3178 | 2·6725 | 3·0657 |

Soln: By data $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$

Divide both sides by 2

$$\frac{2}{2}\frac{d^2y}{dx^2} = \frac{4x + \frac{dy}{dx}}{2}$$

$$\frac{d^2y}{dx^2} = \frac{4x}{2} + \frac{1}{2}\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = 2x + \frac{1}{2}\frac{dy}{dx} \quad \text{—— ①}$$

put $y' = \frac{dy}{dx} = z$

D. w. r. to $x$

$$y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z' \Rightarrow y'' = z'$$

Now eqn ① becomes

$$z' = 2x + \frac{1}{2}z$$

$$z' = 2x + \frac{z}{2}$$

| $x$ | $x_0 = 1$ | $x_1 = 1.1$ | $x_2 = 1.2$ | $x_3 = 1.3$ |
|---|---|---|---|---|
| $y$ | $y_0 = 2$ | $y_1 = 2.2156$ | $y_2 = 2.4649$ | $y_3 = 2.7514$ |
| $y' = Z$ | $Z_0 = 2$ | $Z_1 = 2.3178$ | $Z_2 = 2.6725$ | $Z_3 = 3.0657$ |
| $y'' = Z' = 2x + Z/2$ | $Z_0' = 2x_0 + Z_0/2$ <br> $Z_0' = 3$ | $Z_1' = 2x_1 + Z_1/2$ <br> $Z_1' = 3.3589$ | $Z_2' = 2x_2 + Z_2/2$ <br> $Z_2' = 3.73625$ | $Z_3' = 2x_3 + \frac{Z_3}{2}$ <br> $Z_3' = 4.13285$ |

we first consider milne's predictor formula

$$y_4^P = y_0 + \frac{4h}{3}(2Z_1 - Z_2 + 2Z_3)$$

$$= 2 + \frac{4(0.1)}{3}\left(2(2.3178) - 2.6725 + 2(3.0657)\right)$$

$$= 3.07926$$

$$y_4^{(P)} = 3.0793$$

$$Z_4^{(P)} = Z_0 + \frac{4h}{3}(2Z_1' - Z_2' + 2Z_3')$$

$$= 2 + \frac{4(0.1)}{3}\left(2(3.3589) - (3.73625) + 2(4.13285)\right)$$

$$Z_4^{(P)} = 3.4996$$

$$Z_4' = 2x_4 + Z_4/2$$

$$= 2(1.4) + \frac{3.4996}{2}$$

$$Z_4' = 4.5498$$

Now, we consider milne's corrector formula

$$y_4^{(C)} = y_2 + h/3 (Z_2 + 4Z_3 + Z_4)$$

$$= 2.4649 + \frac{0.1}{3}\left(2.6725 + 4(3.0657) + 3.4996\right)$$

$$= 3.07939$$

$y_4^{(c)} = 3.0794$

$$z_4^{(c)} = z_2 + h/3\,(z_2' + 4z_3' + z_4')$$

$$= 2.6725 + \frac{0.1}{3}(3.73625 + 4(4.13285) + 4.5498)$$

$$z_4^{(c)} = 3.4997$$

Onygain apply corrector formula using appropriate values

$$y_4^{(c)} = y_2 + h/3\,(z_2 + 4z_3 + z_4)$$

$$= 2.4649 + \frac{0.1}{3}(2.6725 + 4(3.0657) + 3.4997)$$

$$y_4^{(c)} = 3.0794$$

④ Do yourself Given the ODE $y'' + xy' + y = 0$ and the following table of initial values, compute $y(0.4)$ by applying Milne's method.

| $x$ | 0 | 0.1 | 0.2 | 0.3 |
|---|---|---|---|---|
| $y$ | 1 | 0.995 | 0.9801 | 0.956 |
| $y'$ | 0 | -0.0995 | -0.196 | -0.2867 |

Solⁿ: By data $y'' + xy' + y = 0$

$$y'' = -xy' - y$$

put $y' = \frac{dy}{dx} = z$

we obtain $y'' = \frac{d^2y}{dx^2} = \frac{dz}{dx} = z'$

$$\therefore y'' = z'$$

given equation becomes

$z' = -xz - y$

$z' = -(xz + y)$

| $x$ | $x_0 = 0$ | $x_1 = 0.1$ | $x_2 = 0.2$ | $x_3 = 0.3$ |
|---|---|---|---|---|
| $y$ | $y_0 = 1$ | $y_1 = 0.995$ | $y_2 = 0.9801$ | $y_3 = 0.956$ |
| $y' = z$ | $z_0 = 0$ | $z_1 = -0.0995$ | $z_2 = -0.196$ | $z_3 = -0.2867$ |
| $y'' = z' = -(xz+y)$ | $z_0' = -1$ | $z_1' = -0.985$ | $z_2' = -0.941$ | $z_3' = -0.87$ |

we first consider milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left(2z_1 - z_2 + 2z_3\right)$$

$$= 1 + \frac{4(0.1)}{3}\left(2(-0.0995) - (-0.196) + 2(-0.2867)\right)$$

$$= \underline{\underline{0.9231}}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}\left(2z_1' - z_2' + 2z_3'\right)$$

$$= 0 + \frac{4(0.1)}{3}\left(2(-0.985) - (-0.941) + 2(-0.87)\right)$$

$$= \underline{\underline{-0.3692}}$$

$$z_4' = -(x_4 z_4 + y_4)$$

$$= -(0.4 \times -0.3692 + 0.9231)$$

$$\underline{\underline{z_4' = -0.7754}}$$

Next we have milne's corrector formula

$$y_4^{(C)} = y_2 + \frac{h}{3}\left(z_2 + 4z_3 + z_4\right)$$

$$= 0.980 + \frac{0.1}{3}\left(-0.196 + 4(-0.2867) + (-0.3692)\right)$$

$$y_4^{(C)} = 0.9229$$

55

$y_4^{(c)} = 0.9229$

$z_4^{(c)} = z_2 + h/3 \left( z_2' + 4z_3' + z_4' \right)$

$= -0.196 + \frac{0.1}{3}\left[-0.941 + 4(-0.87) + (-0.7754)\right]$

$z_4^{(c)} = -0.3692$

Once again put this in corrector formula

$y_4^{(c)} = 0.980 + \frac{0.1}{3}\left[-0.196 + 4(-0.2867) - 0.3692\right]$

$= 0.9229$

Thus $y(0.4) = 0.9229$

(5) Applying milne's predictor and corrector formula compute $y(0.8)$ given that $y$ satisfies the equation $y'' = 2yy'$ and $y$ and $y'$ are governed by the following values.

Do yourself

$y(0) = 0$, $y(0.2) = 0.2027$ $y(0.4) = 0.4228$

$y(0.6) = 0.6841$

$y'(0) = 1$, $y'(0.2) = 1.041$, $y'(0.4) = 1.179$

$y'(0.6) = 1.468$

Apply corrector formula twice

Sol$^n$: Given $y'' = 2yy'$ — (1)

put $y' = z$

D.w.r. to $x$

$y'' = z'$

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$z' = 2yz$

| $x$ | $x_0 = 0$ | $x_1 = 0.2$ | $x_2 = 0.4$ | $x_3 = 0.6$ |
|---|---|---|---|---|
| $y$ | $y_0 = 0$ | $y_1 = 0.2027$ | $y_2 = 0.4228$ | $y_3 = 0.6841$ |
| $y' = z$ | $z_0 = 1$ | $z_1 = 1.041$ | $z_2 = 1.179$ | $z_3 = 1.468$ |
| $y'' = z' = 2yz$ | $z_0' = 0$ | $z_1' = 0.422$ | $z_2' = 0.997$ | $z_3' = 2.009$ |

milne's predictor formula

$$y_4^{(P)} = y_0 + \frac{4h}{3}(2z_1 - z_2 + 2z_3)$$

$$y_4^{(P)} = \underline{1.0237}$$

$$z_4^{(P)} = z_0 + \frac{4h}{3}(2z_1' - z_2' + 2z_3')$$

$$= \underline{2.0307}$$

$$z_4' = 2y_4 z_4$$

$$z_4' = \underline{4.1577}$$

corrector formula

$$y_4^{(C)} = y_2 + \frac{h}{3}(z_2 + 4z_3 + z_4)$$

$$= \underline{1.0282}$$

$$z_4^{(C)} = z_2 + \frac{h}{3}(z_2' + 4z_3' + z_4')$$

$$= \underline{2.0584}$$

applying corrector formula we have

$$y_4^{(C)} = 1.03009$$

Thus $y(0.8) = \underline{1.0301}$

module-05

Continuation

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## Variation of a function

Let us consider a function of $x, y, y'$.
i.e $f(x, y, y') = f(x, y(x), y'(x))$

Suppose we give small increments to $y$ and $y'$ so that they become respectively, $y + h\alpha(x)$, $y' + h\alpha'(x)$ $h$ is small parameter independent of $x$. Now we have

$$f(x, y+h\alpha(x), y'+h\alpha'(x)) = f(x, y, y') + \left(h\alpha\frac{\partial}{\partial y} + h\alpha'\frac{\partial}{\partial y'}\right)f + \frac{1}{2!}\left(h\alpha\frac{\partial}{\partial y} + h\alpha'\frac{\partial}{\partial y'}\right)^2 f + \cdots$$

by using Taylor's expansion.
[$y$ and $y'$ are treated as variables since $x$ is fixed]
Neglecting second and higher degree terms Since $h$ is small parameter, we have

$$f(x, y+h\alpha(x), y'+h\alpha'(x)) - f(x, y, y')$$
$$= h\alpha\frac{\partial f}{\partial y} + h\alpha'\frac{\partial f}{\partial y'}$$

Denoting the LHS of this equation by $\delta f$ we have

$$\delta f = h\alpha\frac{\partial f}{\partial y} + h\alpha'\frac{\partial f}{\partial y'} \quad \text{——} \; (1)$$

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$\delta f$ is called Variation of $f$

we have from ①

$$\delta y = h\alpha \frac{\partial y}{\partial y} + h\alpha' \frac{\partial y}{\partial y'}$$

$$\delta y = h\alpha + h\alpha' \cdot (0)$$

$$\delta y = h\alpha$$

$$\therefore \boxed{h\alpha = \delta y} \quad \text{—— ②}$$

$$\delta y' = h\alpha \frac{\partial y'}{\partial y} + h\alpha' \frac{\partial y'}{\partial y'}$$

$$\delta y' = h\alpha \cdot 0 + h\alpha' (1)$$

$$\delta y' = h\alpha'$$

$$\therefore \boxed{h\alpha' = \delta y'} \quad \text{—— ③}$$

Using ② and ③ in ① we have

$$\delta f = \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y' \quad \text{—— ④}$$

NO NEED

**NOTE:** Geometrically $y(x)$ and $y(x) + h\alpha(x)$ represents two neighbouring curves. Variation in $f$ represents the change in $f$ from curve to curve.

we now proceed to establish two important properties connected with variational operator $\delta$, differential operator $d/dx$ & integral $\int$

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## Property - I

$$\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)$$

proof: $\delta\left(\frac{dy}{dx}\right) = \delta y' = h\alpha'$ by using (3)

$= h\frac{d\alpha}{dx}$

$= \frac{d(h\alpha)}{dx}$ since $h$ is independent of $x$

$= \frac{d}{dx}(\delta y)$ by using (2)

$$\therefore \boxed{\delta\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\delta y)}$$

## Functionals

Let $S$ be a set of functions of a single variable $x$ defined over an interval $(x_1, x_2)$

Then any function which assigns to each function in $S$ a unique real value is called a functional.

In other words functional is a mapping from functions to real numbers.

Consider a function of the form $f(x, y, y')$ where $y'$ is derivative of $y$ w.r.to $x$ and $x \in (x_1, x_2)$

The integral $I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$ is a functional [a standard form]

It can be easily seen that for every $y(x)$, $I(y)$ give a real value.

Examply of functionals

① $\int_0^1 x + (y')^2\, dx$ ② $\int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$

Property - 2

If $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ then

$$\delta \int_{x_1}^{x_2} f(x, y, y')\, dx = \int_{x_1}^{x_2} \delta f(x, y, y')\, dx$$

i.e to say that the variational of a functional associated with $f(x, y, y')$ is equal to the functional associated with variation of $f$.

Proof: $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ is functional

Since the value of $I$ depends on $y$ and $y'$ we have by using the result connected with variation

$$\delta f = \frac{\partial f}{\partial y} \delta y + \frac{\partial f}{\partial y'} \delta y'$$

$\therefore \quad \delta I = \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial y'}\delta y'$

$$\delta I = \left\{\int_{x_1}^{x_2} \frac{\partial}{\partial y}\left[f(x,y,y')\right]dx\right\}\delta y + \left\{\int_{x_1}^{x_2}\frac{\partial}{\partial y'}\left[f(x,y,y')\right]dx\right\}\delta y'$$

i.e $$\delta I = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\delta y\, dx + \int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\delta y'\, dx$$

$$\delta I = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right]dx$$

$$\delta I = \int_{x_1}^{x_2} \delta f\, dx$$

Thus $$\delta\int_{x_1}^{x_2} f(x,y,y')\,dx = \int_{x_1}^{x_2}\delta f(x,y,y')\,dx$$

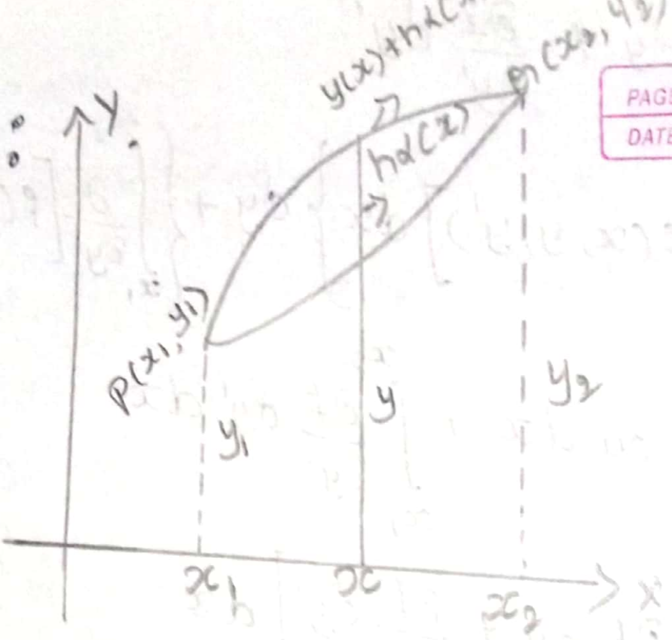
[Dec 17, 18]

## Euler's Equation

Statement: A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x,y,y')\,dx$ where $y(x_1)=y_1$ and $y(x_2)=y_2$ to be an extremum is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad [\text{Euler's equation}]$$

Proof:




Let $I$ be an extremum along some curve $y = y(x)$ passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$

Also, let $y = y(x) + h\alpha(x)$ —— ①

be the neighbouring curve (where $h$ is small) joining these points so that we must have $\alpha(x_1) = 0$ at $P$ and $\alpha(x_2) = 0$ at $Q$ —— ②

when $h = 0$ these two curves coincide thus making $I$ an extremum.

when $h = 0$ these two curves coincide thus making $I$ an extremum.

i.e to say that

$$I = \int_{x_1}^{x_2} f\left(x, y(x) + h\alpha(x), y'(x) + h\alpha'(x)\right) dx$$

Applying chain rule for the partial derivative in RHS, we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h} \right] dx \quad \text{——} ③$$

But $h$ is independent of $x$ and hence $\frac{\partial x}{\partial h} = 0$

Let us consider (1) and D. w. r. to $x$

$\therefore \; y' = y'(x) + h\alpha'(x)$ ——— (4)

Also, we have from (1), $\frac{\partial y}{\partial h} = \alpha(x)$ and

from (4) $\frac{\partial y'}{\partial h} = \alpha'(x)$

using these results in (3) we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x) \right] dx \text{ ——— (5)}$$

Keeping the first term in the RHS of (5) as it is and integrating the second term by parts we have

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x)\,dx + \left\{ \left[ \frac{\partial f}{\partial y'}\alpha(x) \right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx \right\}$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\alpha(x)\,dx + \left\{ \frac{\partial f}{\partial y'}\alpha(x_2) - \frac{\partial f}{\partial y'}\alpha(x_1) \right\}$$
$$- \int_{x_1}^{x_2} \alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

But $\alpha(x_1) = 0 = \alpha(x_2)$ and we have by combining from (2) and the two integrals.

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right] \alpha(x)\,dx$$

But we have already stated that $\frac{dI}{dh}$ must be zero when $h=0$ for $I$ to be an extremum.

hence integrand in the RHS must be zero

Since $\alpha(x)$ is arbitary we must have



$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

This is the required Euler's equation being the necessary condition for the extremum of functional $I = \int_{x_1}^{x_2} f(x, y, y')$

Theorem: The necessary condition for the functional $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ to be an extremum is $\delta I = 0$

Proof: Retrace the steps as in the derivation of Euler's equation upto the stage of arriving at equation (5)

$$\frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\alpha(x) + \frac{\partial f}{\partial y'}\alpha'(x)\right] dx \quad \text{--- (5)}$$

we have $\delta I = \delta \int_{x_1}^{x_2} f(x, y, y')\,dx$

Since $\delta$ and $\int$ are commutative with each other we have

$$\delta I = \int_{x_1}^{x_2} \delta f(x, y, y')\,dx$$

using the expression for the variation of $f$ being $\delta f$ in the RHS, we have

$$\delta I = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'\right] dx$$

But $\delta y = h\alpha(x)$ and $\delta y' = h\alpha'(x)$

$$\delta I = \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} h\alpha(x) + \frac{\partial f}{\partial y'} h\alpha'(x) \right] dx$$

$$\delta I = h \int_{x_1}^{x_2} \left[ \frac{\partial f}{\partial y} \alpha(x) + \frac{\partial f}{\partial y'} \alpha'(x) \right] dx$$

i.e $\delta I = h \frac{dI}{dh}$, by equation (5)

But $\frac{dI}{dh} = 0$ when $h = 0$ is a necessary condition for $I$ to be an extremum.

Thus $\delta I = 0$ also represents the necessary condition for the functional $I$ to be an extremum.

Problems :-

(1) Find the extremal of the functional

$$\int_{x_1}^{x_2} (y' + x^2 y'^2)\, dx$$

(or)

Solve the Euler's equation for the functional $\int_{x_1}^{x_2} (1 + x^2 y') y'\, dx$ [June 2017, 18]

Soln: Let, $f(x, y, y') = y' + x^2 y'^2$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$$0 - \frac{d}{dx}\left(1 + x^2 2y'\right) = 0$$

$$\frac{d}{dx}\left(1 + 2x^2 y'\right) = 0$$

Integrating w.r. to $x$ we get

$1 + 2x^2 y' = K_1$, where $K_1$ is constant

$$2x^2 y' = K_1 - 1$$

$$y' = \frac{K_1 - 1}{2x^2}$$

(or) $$\frac{dy}{dx} = \frac{K_1 - 1}{2x^2}$$

on integration w.r. to $x$

$$y = \frac{K_1 - 1}{2} \int \frac{1}{x^2}\, dx + C_2$$

$$y = \frac{K_1 - 1}{2} \times \frac{-1}{x} + C_2$$

$$y = \frac{1 - K_1}{2x} + C_2$$

$y = \frac{C_1}{x} + C_2$ where $C_1 = \frac{1 - K_1}{2}$

② Find the function $y$ which makes the integral $\int_{x_1}^{x_2} \left(1 + xy' + xy'^2\right) dx$ an extremum.

Solⁿ: Let $f(x, y, y') = 1 + xy' + xy'^2$

Euler's equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0,$$ becomes

$$0 - \frac{d}{dx}\left(x + 2xy'\right) = 0$$

$$\frac{d}{dx}\left(x + 2xy'\right) = 0$$

on integration w.r. to $x$

$x + 2xy' = K_1$, where $K_1$ is constant

$$2xy' = K_1 - x$$

$$y' = \frac{K_1 - x}{2x}$$

or $$\frac{dy}{dx} = \frac{K_1 - x}{2x}$$

$$\frac{dy}{dx} = \frac{K_1}{2x} - \frac{\cancel{x}}{2\cancel{x}}$$

$$\frac{dy}{dx} = \frac{K_1}{2x} - \frac{1}{2}$$

on integration

$$y = \int\left(\frac{K_1}{2x} - \frac{1}{2}\right) dx + C_2$$

$$y = \frac{K_1}{2}\int \frac{1}{x}\, dx - \frac{1}{2}\int 1 \cdot dx + C_2$$

$$y = \frac{K_1}{2}\log x - \frac{1}{2}x + C_2$$

$y = c_1 \log x - \frac{x}{2} + c_2$ where $c_1 = k_1/2$

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③ Find the extremal of the functional

$$\int_{x_1}^{x_2} (y^2 + y'^2 + 2ye^x)\, dx$$

Sol$^n$: Let, $f(x, y, y') = y^2 + y'^2 + 2ye^x$

Euler's equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \text{ becomes}$$

$$(2y + 2e^x) - \frac{d}{dx}(2y') = 0$$

$$\frac{d}{dx}(2y') = 2y + 2e^x$$

$$2\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2y + 2e^x \quad \text{or} \quad 2y'' = 2y + 2e^x$$

$$2 \cdot \frac{d^2y}{dx^2} = 2y + 2e^x \quad \text{or} \quad y'' = y + e^x$$

÷ Both sides by 2

$$\frac{d^2y}{dx^2} = y + e^x$$

$$\frac{d^2y}{dx^2} - y = e^x \Rightarrow D^2y - y = e^x$$

$$(D^2 - 1)y = e^x \quad \text{where } D = \frac{d}{dx}$$

A.E is $m^2 - 1 = 0 \quad \therefore m = \pm 1$

hence CF, $y_c = c_1 e^x + c_2 e^{-x}$

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$$PI = y_p = \frac{\phi(x)}{f(D)}$$

$$y_p = \frac{e^x}{D^2-1}$$

replace D by $-a$ i.e

i.e D by $-1$

$$y_p = \frac{e^x}{(-1)^2-1} = \frac{e^x}{0} \quad (Dr = 0)$$

If Dr is zero diff. Dr and x'ly x to Nr

$$y_p = \frac{x e^x}{2D}$$

$$y_p = \frac{x}{2}\int e^x dx$$

$$y_p = \frac{x}{2} e^x$$

we have $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{-x} + \frac{xe^x}{2}$$

(4) Find the curve on which the following functional $\int_0^1 \left[ (y')^2 + 12xy \right] dx$ with $y(0) = 0$ and $y(1) = 1$ can be determined.

Dec17

Soln: Let $I = \int_0^1 \left[ (y')^2 + 12xy \right] dx$

$$f(x, y, y') = (y')^2 + 12xy$$

Euler's equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$12x - \frac{d}{dx}(2y') = 0 \quad \text{or} \quad 12x - \frac{d}{dx}(2y') = 0$$

$$12x - 2y'' = 0$$

$$12x - \frac{d}{dx}\left(2 \cdot \frac{dy}{dx}\right) = 0$$

$$12x - 2 \cdot \frac{d^2y}{dx^2} = 0$$

$$12x = 2\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = 6x$$ and integrating w.r. to $x$

we get

$$\frac{dy}{dx} = \frac{6x^2}{2} + c_1$$

$$\frac{dy}{dx} = 3x^2 + c_1$$

Again integrating w.r. to $x$

$$y = \frac{3x^3}{3} + c_1 x + c_2$$

$$y = x^3 + c_1 x + c_2 \quad \text{—} \ (*)$$

Using the condition $y=0$ at $x=0$ in $(*)$

$$0 = 0 + 0 + c_2 \Rightarrow c_2 = 0$$

and we $y=1$ at $x=1$

$$1 = 1 + c_1 + 0$$

$$c_1 = 1 - 1 \Rightarrow \underline{\underline{c_1 = 0}}$$

put $c_1$ and $c_2$ in ⊛

$y = x^3$ is the required curve

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(5) Solve the Variational problem

$\delta \int_0^1 (12xy + y'^2)\, dx$ under the conditions

Dec 18

$y(0) = 3$ and $y(1) = 6$

Let $I = \int_0^1 (12xy + y'^2)\, dx$, $\delta I = 0$ is equivalent to Euler's eqn.

Solⁿ: $f(x, y, y') = 12xy + (y')^2$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$12x - \frac{d}{dx}(2y') = 0$$

$$12x - 2\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad 12x - 2y'' = 0$$

$$\frac{d^2y}{dx^2} = 6x$$

Integrating w.r.to $x$ we get

$$\frac{dy}{dx} = \frac{6x^2}{2} + c_1$$

$$\frac{dy}{dx} = 3x^2 + c_1$$

Again integrating w.r. to $x$

$$y = x^3 + c_1 x + c_2 \quad \text{— ⊛}$$

use the condition $y = 3$ at $x = 0$ in ⊛

$$3 = 0 + 0 + c_2 \Rightarrow c_2 = 3$$

put $x = 1$, $y = 6$ in ⊛

$$6 = 1 + c_1 + 3 \Rightarrow c_1 = 6 - 4 = 2$$

$$c_1 = 2$$

put $c_1$ and $c_2$ in (*)

$y = x^3 + 2x + 3$ is the required curve

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Do yourself

* Solve the Variational problem:

$\delta \int_0^1 (x + y + y'^2)\, dx = 0$ under the conditions $y(0) = 1$ and $y(1) = 2$.

Ans: $y = \frac{x^2}{4} + \frac{3}{4}x + 1$ $\quad c_2 = 1$

$c_1 = 3/4$

(*6) S.T the functional $\int_{x_1}^{x_2} (y^2 + x^2 y')\, dx$ assumy extreme valuy on the straight line $y = x$

Soln: Let, $f(x, y, y') = y^2 + x^2 y'$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$2y - \frac{d}{dx}(x^2) = 0$$

$$\frac{d}{dx}(x^2) = 2y$$

$$\cancel{2}x = \cancel{2}y$$

$x = y$ or $y = x$ is straight line

(7) S.T $\int_{x_1}^{x_2} y^2 y'^2 \, dx$ has an extremum when $y(x)$ is of the form $c_1 \cdot \sqrt{x + c_2}$

Sol: Let $f(x, y, y') = y^2 y'^2$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$2yy'^2 - \frac{d}{dx}(2y^2y') = 0$$

$$yy'^2 - \frac{d}{dx}(y^2y') = 0$$

$$yy'^2 - (y^2y'' + 2yy'y') = 0$$

$$yy'^2 - (y^2y'' + 2yy'^2) = 0$$

$$-y^2y'' - yy'^2 = 0$$

$$y^2y'' + yy'^2 = 0$$

$$y[yy'' + y'^2] = 0 \quad (\div \text{B.S by } y)$$

$$yy'' + y'^2 = 0$$

$$\frac{d}{dx}[yy'] = 0$$

on integration we get

$$yy' = K_1$$

$$y\frac{dy}{dx} = K_1$$

$$y\,dy = K_1\,dx$$

$$\int y\,dy = K_1 \int 1\,dx + K_2$$

$$\frac{y^2}{2} = K_1 x + K_2$$

$$y^2 = 2(K_1 x + K_2)$$

$$y = \sqrt{2(K_1 x + K_2)}$$

$$y = \sqrt{2K_1\left(x + K_2/K_1\right)}$$

let us denote $c_1 = \sqrt{2K_1}$ and $c_2 = K_2/K_1$

Thus $y = c_1\sqrt{x + c_2}$

(8) S.T an extremal of $\int_{x_1}^{x_2} \left(\frac{y'}{y}\right)^2 dx$ is expressible in the form $y = ae^{bx}$

Sol$^n$: Let $f(x, y, y') = \left(\frac{y'}{y}\right)^2$

$$f(x, y, y') = \frac{y'^2}{y^2}$$

Euler's equation, $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$-\frac{2}{y^3} y'^2 - \frac{d}{dx}\left(\frac{1}{y^2} \times 2y'\right) = 0$$

$$-2\left[+\frac{y'^2}{y^3} + \frac{d}{dx}\left(\frac{y'}{y^2}\right)\right] = 0$$

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÷ B.S by $-2$

$$\frac{y'^2}{y^3}+\frac{d}{dx}\left(\frac{y'}{y^2}\right)=0$$

$$\frac{y'^2}{y^3}+\frac{y^2y''-2yy'y'}{y^4}=0$$

$$\frac{y'^2}{y^3}+\frac{y^2y''-2yy'^2}{y^4}=0$$

$$\frac{y'^2}{y^3}+\frac{\cancel{y}(yy''-2y'^2)}{\cancel{y^4}}=0$$

$$\frac{y'^2}{y^3}+\frac{yy''-2y'^2}{y^3}=0$$

$$\frac{y'^2+yy''-2y'^2}{y^3}=0$$

$$y'^2+yy''-2y'^2=0$$

$$yy''-y'^2=0$$

Now it can be put in the form

$\frac{d}{dx}\left(\frac{y'}{y}\right)=0$ on integration w.r. to $x$

$$\frac{y'}{y}=C_1$$

again integrate

hence, $\int \frac{y'}{y} dx = \int c_1 dx + c_2$

$\log_e y = c_1 x + c_2$

$y = e^{c_1 x + c_2}$

$y = e^{c_1 x} \cdot e^{c_2}$

$y = ae^{bx}$ where $a = e^{c_2}$ and $b = c_1$

(9) Find the curve on which the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)dx$ with $y(0) = y(\pi/2) = 0$ can be extremized

Sol$^n$: Let $f(x, y, y') = y'^2 - y^2 + 2xy$

Eulers equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$-2y + 2x - \frac{d}{dx}(2y') = 0$$

$$-y + x - \frac{d}{dx}(y') = 0$$

$$-y + x - y'' = 0$$

$$y'' + y = x$$

$$\frac{d^2y}{dx^2} + y = x \Rightarrow D^2y + y = x$$

$$(D^2+1)y = x$$

where $D = d/dx$

A.E is $m^2+1=0 \Rightarrow m^2 = -1 \Rightarrow m = \pm\sqrt{-1}$

$m = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

$y_p = \dfrac{\phi(x)}{f(D)}$

$y_p = \dfrac{x}{D^2+1}$

$$\begin{array}{r|l} & x \\ \hline 1+D^2 & x \\ & -x \\ \hline & 0 \end{array}$$

$y_p = \dfrac{x}{1+D^2}$

$y_p = x$

$y = y_c + y_p$

$\underline{\underline{y = c_1 \cos x + c_2 \sin x + x}}$ — (*)

use the given conditions $y(0) = y(\pi/2) = 0$

i.e $y(0) = 0$ and $y(\pi/2) = 0$

put $x=0$ & $y=0$ in (*)

$0 = c_1 \cos(0) + c_2 \sin(0) + 0$

$0 = c_1 \Rightarrow \underline{\underline{c_1 = 0}}$

put $y=0$ and $x = \pi/2$ in (*)

$0 = c_1 \cos \pi/2 + c_2 \sin \pi/2 + \pi/2$

$0 = (0)(0) + c_2(1) + \pi/2$

$C_2 = -\pi/2$

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put $C_1$ and $C_2$ in ⊛

$y = (0)\cos x + \sin x(-\pi/2) + x$

$y = -\pi/2 \sin x + x$ is the required curve

## Geodesics

A geodesic on a surface is a curve along which the distance between any two points of the surface is minimum

## Standard variational problems

① P.T the shortest distance between two points in a plane is along the straight line joining them or Prove that geodesics on a plane are straight lines.

June17, Dec16, 18

Soln: Let $y = y(x)$ be a curve joining two points ~~in a plane is along the~~, $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the xoy plane.

W.K.T the arc length between P and Q is given by

$$S = \int_{x_1}^{x_2} \frac{ds}{dx}\, dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

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$$S = I = \int_{x_1}^{x_2} \sqrt{1+(y')^2}\, dx$$

we need to find the curve $y(x)$ such that $I$ is minimum

Let, $f(x, y, y') = \sqrt{1+(y')^2}$

Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ becomes

$$0 - \frac{d}{dx}\left[\frac{1}{\cancel{2}\sqrt{1+(y')^2}} \times \cancel{2}y'\right] = 0$$

$$\frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right] = 0 \quad (\text{apply } u/v \text{ rule})$$

$$\frac{y''\sqrt{1+(y')^2} - y'\frac{1}{\cancel{2}\sqrt{1+(y')^2}} \times \cancel{2}y'y''}{\left(\sqrt{1+(y')^2}\right)^2} = 0$$

$$y''\sqrt{1+(y')^2} - (y')^2 y'' \frac{1}{\sqrt{1+(y')^2}} = 0$$

$$\frac{y''\left(\sqrt{1+(y')^2}\right)^2 - (y')^2 y''}{\sqrt{1+(y')^2}} = 0$$

$$y''\left(\cancel{\sqrt{\ }}\,1+(y')^2\right)^2 - y''(y')^2 = 0$$

$$y''(1+(y')^2) - y''(y')^2 = 0$$

$$y'' + \cancel{y''(y')^2} - \cancel{y''(y')^2} = 0$$

$$y'' = 0$$

$$\frac{d^2y}{dx^2} = 0$$

Integrate w.r. to $x$

$$\frac{dy}{dx} = c_1$$

again integrate w.r. to $x$

$y = c_1 x + c_2$ which is a straight line

② Find geodesics on a surface given that arc length on the surface is $S = \int_{x_1}^{x_2} \sqrt{x(1+(y')^2)}\, dx$

June 2018

Soln: we have $f = \sqrt{x(1+(y')^2)}$

which is independent of $y$

$\therefore$ Euler's equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$0 - \frac{d}{dx}\left(\frac{1}{\cancel{2}\sqrt{x(1+(y')^2)}} \times x\,\cancel{2}y'\right) = 0$$

$$\frac{d}{dx}\left(\frac{xy'}{\sqrt{x(1+(y')^2)}}\right) = 0$$

Integrate w.r. to $x$

$$\frac{xy'}{\sqrt{x(1+(y')^2)}} = C$$

$$\frac{xy'}{\sqrt{x}\sqrt{1+(y')^2}} = C$$

$$\left\{ \frac{x}{\sqrt{x}} = \frac{\sqrt{x}\sqrt{x}}{\sqrt{x}} = \sqrt{x} \right.$$

$$\sqrt{x}\,\frac{y'}{\sqrt{1+(y')^2}} = C$$

$$\sqrt{x}\,y' = C\sqrt{1+(y')^2}$$

S. on B.S

$$x(y')^2 = c^2(1+(y')^2)$$

$$x(y')^2 = c^2 + c^2(y')^2$$

$$x(y')^2 - c^2(y')^2 = c^2$$

$$(y')^2(x-c^2) = c^2$$

$$(y')^2 = \frac{c^2}{x-c^2}$$

$$y' = \sqrt{\frac{c^2}{x-c^2}}$$

$$y' = \frac{c}{\sqrt{x-c^2}} \quad \text{or} \quad \frac{dy}{dx} = \frac{c}{\sqrt{x-c^2}}$$

integrate w.r. to $x$

$$y = c\int \frac{1}{\sqrt{x-c^2}}\,dx + c_1$$

$$y = 2c\sqrt{x-c^2} + c_1$$

$$y - c_1 = 2c\sqrt{x-c^2}$$

S. on. B. S

$$(y-c_1)^2 = 2^2c^2(x-c^2)$$

$(y-c_1)^2 = 4c^2(x-c^2)$ is the required geodesic which is a parabola.

③ P.T Catenary is the curve which when rotated about a line generate a Surface of minimum area.

Do yourself

Soln: we have the expression for the total surface area given by $\int 2\pi y\, ds$ where the curve is rotated about x-axis.

$$\therefore I = \int_{x_1}^{x_2} 2\pi y \frac{ds}{dx}\, dx$$

$$= \int_{x_1}^{x_2} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{x_1}^{x_2} 2\pi y \sqrt{1+(y')^2}\, dx$$

Since $2\pi$ is constant we can as well take $f(x,y,y') = y\sqrt{1+(y')^2}$ which is independent of $x$.

$\therefore$ It is convenient to take the Euler's equation in the form.

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

i.e, $$y\sqrt{1+(y')^2} - y' \cdot \frac{y}{\cancel{2}\sqrt{1+(y')^2}}\cancel{2}y' = c$$

$$\frac{y\left(\sqrt{1+(y')^2}\right)^2 - (y')^2 y}{\sqrt{1+(y')^2}} = c$$

$$y(1+(y')^2) - y(y')^2 = c\sqrt{1+(y')^2}$$

$$y + \cancel{y(y')^2} - \cancel{y(y')^2} = c\sqrt{1+(y')^2}$$

$$y = c\sqrt{1+(y')^2}$$

S. on B. S

$$y^2 = c^2\left(\sqrt{1+(y')^2}\right)^2$$

$$y^2 = c^2(1+(y')^2)$$

$$y^2 = c^2 + c^2(y')^2$$

$$c^2(y')^2 = y^2 - c^2 \Rightarrow (y')^2 = \frac{y^2-c^2}{c^2}$$

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$$y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\,dx$$

$$\int \frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\int dx + k$$

$$\cosh^{-1}(y/c) = \frac{x}{c} + k$$

$$y/c = \cosh(x/c + k)$$

$$y = c\cosh(x/c + k)$$

$$y = c\cosh\left(\frac{x+ck}{c}\right)$$

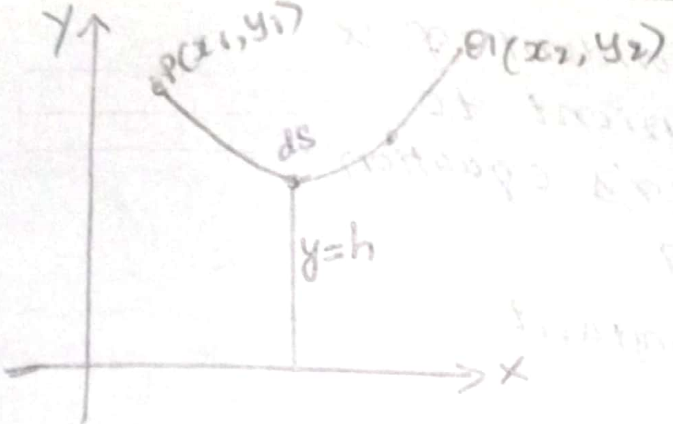
$$y = c\cosh\left(\frac{x+a}{c}\right)$$ where $a = kc$, this is a

Catenary

* Dec 16, 18    Hanging cable (chain) problem

* A heavy cable hangs freely under gravity between two fixed points S.T the shape of the cable is a catenary.

Sol^n^






Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two fixed points of the hanging cable. Let us consider an elementary arc length $ds$ of the cable. Let $\rho$ be the density (mass/unit length) of the cable so that $\rho\, ds$ is the mass of the element. If $g$ is the acceleration due to gravity then the potential energy of the element $(m.g.h)$ is given by $(\rho\, ds).g.y$ where x-axis is taken as the line of refrence.

$\therefore$ Total potential energy of the cable is given by

$$T = \int_P^Q (\rho\, ds) \cdot g y \, dx$$

$$= \int_{x_1}^{x_2} \rho g y \frac{ds}{dx} \, dx$$

But $\dfrac{ds}{dx} = \sqrt{1 + (y')^2}$

here, $f(x, y, y') = (\rho g) y \sqrt{1 + (y')^2}$

$$= \text{const.}\ y\sqrt{1 + (y')^2}$$

which is independent of $x$

$\therefore$ It is convenient to take the Euler's equation in the form

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$$f - y'\frac{\partial f}{\partial y'} = \text{constant}$$

$$y\sqrt{1+(y')^2} - y'\frac{y}{\cancel{2}\sqrt{1+(y')^2}} \times \cancel{2}y' = C$$

$$y\sqrt{1+(y')^2} - \frac{(y')^2 y}{\sqrt{1+(y')^2}} = C$$

$$\frac{y(\sqrt{1+(y')^2})^2 - (y')^2 y}{\sqrt{1+(y')^2}} = C$$

$$y(1+(y')^2) - y(y')^2 = C\sqrt{1+(y')^2}$$

$$y + \cancel{y(y')^2} - \cancel{y(y')^2} = C\sqrt{1+(y')^2}$$

$$y = C\sqrt{1+(y')^2}$$

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$$y^2 = C^2(1+(y')^2)$$

$$y^2 = c^2 + c^2(y')^2$$

$$c^2(y')^2 = y^2 - c^2$$

$$(y')^2 = \frac{y^2-c^2}{c^2}$$

$$y' = \sqrt{\frac{y^2-c^2}{c^2}}$$

$$y' = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-c^2}}{c}$$

$$\frac{dy}{\sqrt{y^2-c^2}} = \frac{1}{c}\,dx$$

$$\int \frac{1}{\sqrt{y^2-c^2}}\,dy = \frac{1}{c}\int dx + K$$

$$\cosh^{-1}(y/c) = x/c + K$$

$$y/c = \cosh(x/c + K)$$

$$y = c\cosh(x/c + K)$$

$$y = c\cosh\left(\frac{x+cK}{c}\right)$$

$$y = c\cosh\left(\frac{x+a}{c}\right) \quad \text{where } a = Kc$$

which is catenary